

# DIGITAL FILTER FOR QUADRATURE SUB-SAMPLE DELAY ESTIMATION

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## ABSTRACT

*In this paper we present a novel digital filter intended for quadrature sub-sample delay estimation based on the Hilbert (complex) filtering and fractional delaying of an incoming real-valued signal. It is a Hilbert Transform Filter (HTF) having variable fractional delay (VFD). The filter is based on a pair of rotated VFD filters in the Farrow structure. It is capable of performing the Hilbertian as well as VFD filtering of the incoming discrete-time signal at the same time. Thus one can substitute a hitherto used cascade of the HTF and the VFD filters with an aggregated HTF-VFD filter proposed here. The technique is simple to implement. The advantages lie in lower total delay introduced by the compound filter and in modular structure, which is composed of the same linear-phase rotated VFD Farrow sub-filters. The overall rotated VFD filters in the pair differ only in the value of one parameter – the VFD. The proposed filter can be applied to adaptive quadrature sub-sample estimation of delay.*

## I. INTRODUCTION

Fractional delay filters (FDFs) find a variety of applications in, e.g., sound synthesis, timing adjustment in digital receivers, image zooming and many others, which have been widely described so far see, e.g., [12]. Recently a FDF having variable fractional delay (VFD) has been applied to adaptive sub-sample delay estimation using a quadrature detector based on the Hilbert (complex) filtering and fractional delaying of an incoming real-valued signal [1], [2], [3]. In the abovementioned references both of these operations have been realized by two independent filters, namely the Hilbert transform filter (HTF) and the VFD filter. In this paper we propose to substitute these filters with a single aggregated one, further called the HTF-VFD, having the same functionality, but reduced transport delay. For this aim we use polynomial interpolation. The polynomial interpolation can be implemented efficiently using the idea based on the Farrow structure [4], [5]. This structure is very attractive because it can provide variable signal delay by changing only one parameter. This makes on-line delaying straightforward and feasible. The signal passes through FIR Farrow subfilters either symmetric or anti-symmetric see [4], hence their phase response linearity, and is multiplied by appropriate powers of delay to produce the output.

Organization of the paper is the following. Section II provides indispensable background information about the fractional delay filter and the Farrow structure. Section III

formulates the procedure of designing the HTF having fractional delay. Section IV provides illustrative design examples. Section V summarizes the results.

## II. BACKGROUND

### Fractional Delay Filter

The frequency response of an ideal FDF is defined as

$$D_{\tau}(e^{j\omega}) = \exp(-j\omega\tau), \quad |\omega| < \pi \quad (1)$$

where  $\omega = 2\pi FT$  is the normalized angular frequency in rad per sample,  $F$  stands for the physical frequency in Hz and  $T$  stands for the sample interval in seconds. The impulse response of this filter is

$$\begin{aligned} d_{\tau}[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_{\tau}(e^{j\omega}) e^{j\omega n} d\omega = \\ &= \frac{\sin \pi(n-\tau)}{\pi(n-\tau)} = \text{sinc}(n-\tau) \end{aligned} \quad (2)$$

In the case of an FIR approximation of length  $N$ ,  $\tau$  is defined as a sum of a transport delay of the system and introduced fractional delay  $\alpha$

$$\tau = (N-1)/2 + \alpha \quad (3)$$

where the values of  $\alpha$  are restricted to the interval  $-0.5 \leq \alpha \leq 0.5$  in order to deal with central interpolation – the most accurate one.

The transfer function of an FIR FDF in a direct form is defined as

$$D_{N\alpha}(z) = \sum_{n=0}^{N-1} d_{N\alpha}[n] z^{-n} \quad (4)$$

where  $d_{N\alpha}[n]$ ,  $n = 0, 1, \dots, N-1$  stands for the impulse response of the FIR filter. This filter, in case of variable value of  $\alpha$ , is called the VFD.

### Farrow Structure

High efficiency of a VFD can be achieved by applying the Farrow structure [4]. The main advantages of this structure is that the coefficients of its subfilters are either symmetrical or anti-symmetrical and fixed, independent of the value of introduced fractional delay  $\alpha$  see [4]. It means that we can change the fractional delay  $\alpha$  without redesigning the filter.

In order to apply the Farrow structure we have to express the equation (4) in a form of a polynomial of fractional delay  $\alpha$ . The coefficients of the filter can be rewritten as

$$d_{N\alpha}[n] = \sum_{k=0}^M c_k[n] \alpha^k, \quad n = 0, 1, \dots, N-1 \quad (5)$$

where  $M+1$  stands for a number of subfilters in the Farrow structure.

From the above equations we have

$$D_{N\alpha}(z) = \sum_{k=0}^M \sum_{n=0}^{N-1} c_k[n] z^{-n} \alpha^k = \sum_{k=0}^M C_k(z) \alpha^k \quad (6)$$

where

$$C_k(z) = \sum_{n=0}^{N-1} c_k[n] z^{-n}, \quad k = 0, 1, \dots, M \quad (7)$$

are the transfer functions of subfilters, which in the original Farrow work [4] are implemented in direct form. This structure gives us a flexibility to change the delay by manipulating only one coefficient, the aforementioned fractional delay (FD).

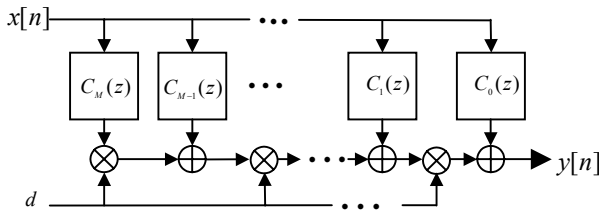


Fig. 1. VFD in the Farrow structure;  $y[n] = x(n - (N-1)/2 - \alpha)$  and  $d$  is given by (10b). The subfilters are linear-phase [4].

### III. FILTER DESIGN AND IMPLEMENTATION

#### Hilbert Transform Filter

The frequency response of an ideal Hilbert transform filter having the so-called generalized linear phase-response is defined [6] as

$$H_\beta(e^{j\omega}) = \begin{cases} 2 \exp(-j(\omega - \pi/2)\beta), & \omega \in (0, \pi) \\ 0, & \omega \in (-\pi, 0) \end{cases} \quad (8)$$

where  $\beta$  stands for the delay introduced by the filter. We can write the impulse response of this filter using a pair of VFDs [6]

$$h_\beta[n] = \begin{cases} (-1)^{n/2} d_{\beta/2}[n/2], & n = 0, \pm 2, \pm 4, \dots \\ (-1)^{n/2} d_{(\beta-1)/2}[(n-1)/2], & n = \pm 1, \pm 3, \dots \end{cases} \quad (9)$$

Therefore the design of an FIR approximation of length  $2N$  for this ideal HTF requires two VFDs of length  $N$  each.

A general scheme of implementing this HTF having variable fractional delay as an FIR filter in the Farrow structure is presented in Fig. 2. The total delay of this filter expressed in terms of  $\alpha$  is

$$\beta = N - 1 + 2\alpha \quad (10a)$$

and the fractional delay to be varied is

$$d = 2\alpha - 1/2 \quad (10b)$$

In Fig. 2 in the lower branch we need to introduce an additional delay  $z^{-1}$  necessary to interlace the coefficients of the filter.

In case of Lagrangian polynomial approximation the coefficients of a maximally flat VFD can be obtained using the following expression

$$d_{N\alpha}[n] = \prod_{\substack{k=0 \\ k \neq n}}^{N-1} \frac{\alpha - k}{n - k} \quad (11)$$

for  $n = 0, 1, \dots, N-1$ , where  $\alpha = (\beta - N + 1)/2$  from (10a).

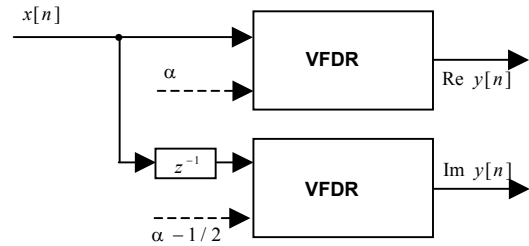


Fig. 2. Block scheme for implementing the HTF-VFD in one stage with variable fractional delay  $d = 2\alpha - 1/2$  (10b).

The procedure of designing the filter in Fig. 2 is as follows. From equation (11) we compute the coefficients of the VFD in direct form and then, using equations (4), (6) and (7), we compute the coefficients  $\{c_k[n]\}_{k=0}^{N-1}$ . The next step is multiplication of these coefficients by appropriate powers of  $(-1)$ , as given by

$$\{\tilde{c}_k[n]\}_{k=0}^{N-1} = \{(-1)^n c_k[n]\}_{k=0}^{N-1} \quad (12)$$

in order to get their “rotated” version (cf. (9)). The resulting filter is called in Fig. 2 the VFDR – variable fractional delayer “rotated”. Noteworthy is that we introduce the delay  $\alpha$  into the upper branch in Fig. 2 and  $\alpha - 1/2$  into the lower branch. The output of the HTF-VFD is a complex signal,  $y[n]$ , described as

$$y[n] = \text{Re } y[n] + j \text{Im } y[n] \quad (13)$$

The transfer function of the resulting HTF-VFD having variable fractional delay  $2\alpha$  is

$$H_{\text{HTF-VFD}}(z) = \sum_{k=0}^{N-1} \tilde{C}_k(z^2) \alpha^k + j z^{-1} \sum_{k=0}^{N-1} \tilde{C}_k(z^2) (\alpha - \frac{1}{2})^k$$

### IV. DESIGN EXAMPLES

In this Section we present some examples of designed HTF-VFD filters.

#### A. Maximally flat HTF-VFD

**Example 1.** The coefficients of Farrow subfilters with a maximally flat VFDR of length  $N=2$  for the HTF-VFD from Fig. 2 of length 4, computed in accordance with the description as above are the following

$$\{\tilde{c}_0[n]\}_{n=0}^1 = \{1/2, -1/2\} \quad \text{and} \quad \{\tilde{c}_1[n]\}_{n=0}^1 = \{-1, -1\}.$$

Consequently

$$\tilde{C}_0(z) = 1/2 - 1/2z^{-1} \quad \text{and} \quad \tilde{C}_1(z) = -1 - z^{-1},$$

and

$$H_{\text{HTF-VFD}}(z) = \sum_{k=0}^1 \tilde{C}_k(z^2) \alpha^k + jz^{-1} \sum_{k=0}^1 \tilde{C}_k(z^2) (\alpha - \frac{1}{2})^k$$

$$= 1/2 - \alpha + j(1-\alpha)z^{-1} - (1/2 + \alpha)z^{-2} - j\alpha z^{-3}$$

The impulse response and the coefficients of  $H_{\text{HTF-VFD}}(z)$  for an arbitrary value of  $N$  for this aggregated filter and also for the corresponding cascade can be readily obtained using the symbolic function `HTF-VFD_symb(N)` written in MATLAB and given below, where small “d” plays the role of “ $\alpha$ ” in (14). An exemplary display of results for  $N=2$  is also provided.

```
function [htr,htrcascade]=HTF-VFD_symb(N);
%
% program function for the generation of the coefficients of
% FIR fractional delaying Hilbert transform filter
% HTF-VFD maximally flat around omega=pi/2
% using symbolic expressions for the impulse response
% coefficients htr(n)
% as a function of FD d; it is assumed that -0.5<=d<=0.5
% input
% N - impulse response length of the VFD solely
% output
% htr - the aggregated HTF-VFD coefficients vector of d
% htrcascade - the HTF and VFD cascade imp. resp. of d
% use [htr,htrcascade]=HTF-VFD_symb(2); to start with N=2
syms d h1 h2 h11 h21 htr hfsd hfsd1 hfsd2 htotal
syms htrafo hfsd sum hd g
% HTF-VFD aggregated
h1=fdfsymb(N); h2=fdfsymb_shifted(N);
for n=1:N; h11(n)=((-1)^(n-1))*h1(n); h21(n)=j*(-1)^(n-1)*h2(n); end
htr(1:2*N)=0; for n=1:N; htr(2*n-1)=h11(n); htr(2*n)=h21(n); end
% Cascade of linear-phase HTF and VFD
% Linear-phase HTF
hupper=vfd(N,0.5); hlower=vfd(N,0);
for n=1:N; h11upper(n)=((-1)^(n-1))*hupper(n);
h21lower(n)=j*(-1)^(n-1)*hlower(n); end
h0(1:2*N)=0; for n=1:N; h0(2*n-1)=h11upper(n);
h0(2*n)=h21lower(n); end
htrafo(1:2*N-1)=h0(2:2*N);
hfsd1=fdfsymb(N); % VFD
for n=1:2*N-1; hfsd2(n)=0; end;
for n=1:N; hfsd2(2*n-1)=hfsd1(n); end
for n=1:2:2*N-1; hfsd(n)=imag((-1)^(n/2))*hfsd2(n); end;
% Cascade
% convolution of htrafo and hfsd is computed symbolically
Nx=2*N-1; hd=htrafo; g=hfsd; sum(1)=hd(1)*g(1);
for n=2:Nx; sum(n)=hd(1)*g(n);
for k=2:n; sum(n)=hd(k)*g(n-k+1)+sum(n); end; end
sum(2*Nx-1)=hd(Nx)*g(Nx);
for n=2*Nx-2:-1:Nx+1; p=1; sum(n)=hd(Nx)*g(Nx-p);
for k=Nx:-1:n-(Nx-1); sum(n)=hd(k)*g(n-k+1)+sum(n); end;
p=p+1; end; htrcascade=sum;
```

```
HTF-VFD_imp_resp=conj(htr')
HTF-VFD_cascade_imp_resp=conj(htrcascade')
```

```
function h=fdfsymb(N)
syms d h h1; tr=(N-1)/2; for n=1:N; h(n)=1;
for k=1:N; if k~=n h(n)=h(n)*(tr+d-(k-1))/(n-k); end; end; end
```

```
function h=fdfsymb_shifted(N)
syms d h h1; tr=(N-1)/2;%transport delay
for n=1:N; h(n)=1;
for k=1:N; if k~=n h(n)=h(n)*(tr+d-1/2-(k-1))/(n-k); end; end; end
```

```
function h=vfd(N,de)
taug=(N-1)/2+de; for n=1:N; h(n)=1;
for k=1:N; if k~=n h(n)=h(n)*(taug-(k-1))/(n-k); end; end; end
```

The display of results for  $N=2$

```
HTF-VFD_imp_resp =
[ 1/2-d]; [ i*(-d+1)];
[ -1/2-d]; [ -i*d];

HTF-VFD_cascade_imp_resp =
[ 1/2*i*(1/2-d)]; [ -1/2+d];
[ -1/2*i*(1/2-d)+1/2*i*(-1/2-d)]; [ 1/2+d];
[ -1/2*i*(-1/2-d)];
```

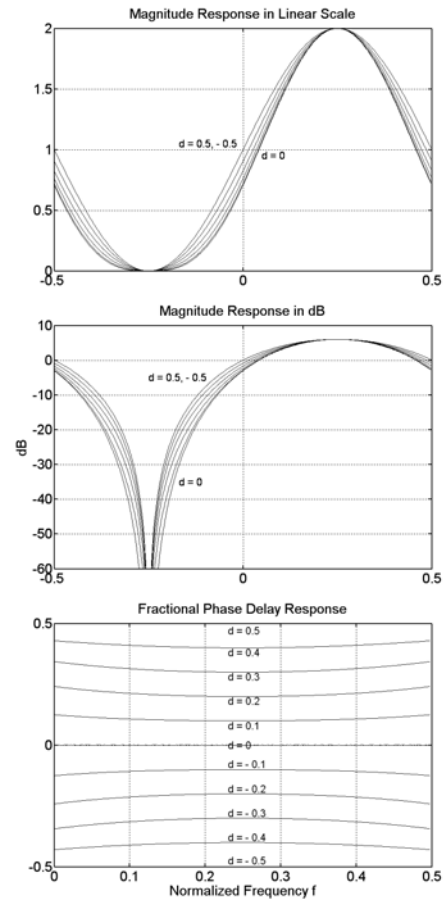


Fig. 3. Continued on the next page.

Fig. 3 presents the performance of the designed narrowband aggregated filter, where  $d$  is defined in (10b).

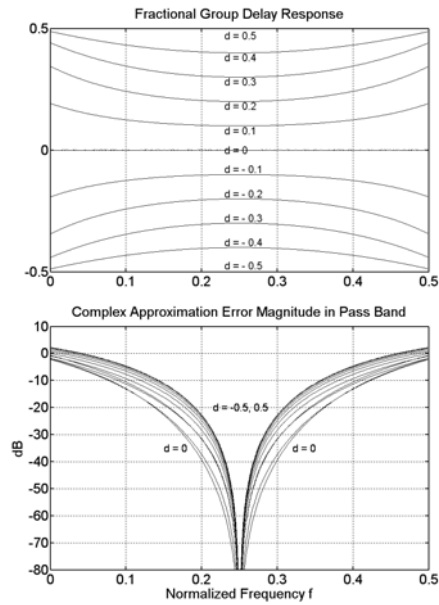


Fig. 3. The performance of an aggregated HTF-VFD of length 4, maximally flat at  $f = \pi/2$ , from Example 1.

For phase and group delay responses in Fig. 3 the transport delay having the value of 1.5 sample intervals is removed for better visualization of the FD. The FD values are set here from the region  $\alpha \in <0, 0.5>$  for the upper branch of Fig. 2 and  $\alpha - 1/2 \in <-0.5, 0>$  for the lower branch, with an increment equal to 0.1. It finally gives the VFD value  $d = 2\alpha - 1/2 \in <-0.5, 0.5>$  as depicted in Fig. 3. It is worth noticing that magnitude responses are displayed in the range  $f \in <-1/2, 1/2>$ , where  $f = \omega/(2\pi)$ , on the normalized frequency axis and other responses are displayed within the range  $f \in <0, 1/2>$ . The latter interval covers the pass band of the filter, where these characteristics are the object of our interests and neglects the stop band, where only the attenuation is important.

For the sake of comparison Fig. 4 presents the performance of the corresponding cascade filter see [1], [2] or [3], for the same FD delay values as in Example 1. The target filter length is 5. The transport delay introduced by the cascade is 2 sample intervals, thus is greater than in Example 1. Also the width of pass band is unfavorably narrower than for the aggregated design proposed here. Thus in this example we have shown that the novel HTF-VFD filter from Fig. 3 performs better than its hitherto cascade counterpart from Fig. 4, both based on the same VFD module. This holds independently of the VFD module length.

#### B. Minimax HTF-VFD optimized in two-stage design

Finally, in Fig. 5, we present the performance of a wideband HTF-VFD with a pair of minimax VFDs optimized in two-stage design [7], [8]. Two-stage design

means that at the front end of processing a linear-phase half-band filter (HBF) produces two-times over-sampled signal which is passed through a half-band VFD. The result is finally down-sampled back to the base-band frequency. In this arrangement the requirements for the VFD are relaxed relative to one-stage approach allowing for a smaller number of taps in the Farrow structure. In [7] it was shown that the two-rate structure could be realized efficiently as two-stage one-rate with two parallel branches.

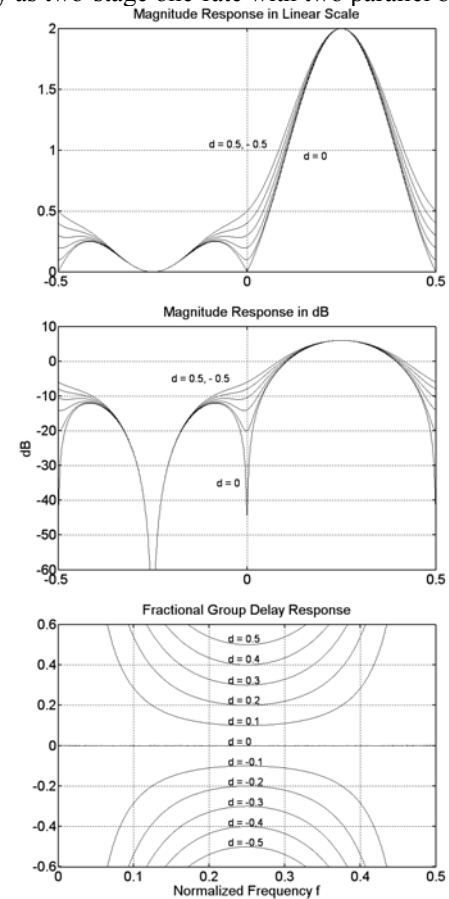


Fig. 4. The performance of a cascade of maximally flat HTF and VFD.

The specification on VFDs given in [7] is the following. The bandwidth is 0.45 expressed in terms of the normalized frequency  $f$ . The maximal absolute value of the complex approximation error (CAE) magnitude defined as the difference between the ideal frequency response (1) and its minimax approximate is 0.0042 which is equivalent to  $-47.5$  dB. The coefficients of Farrow VFD subfilters rounded to four decimal digits are given in Figure 4 in [7]. The number of these subfilters as well as their length is jointly optimized [7]. The HBF is a Nyquist filter [11] of order 58, optimal in the Chebyshev sense, with pass band and stop band edge normalized frequencies: 0.45 and 0.55, respectively, requiring 15 taps of different values. The overall VFD needs 14 taps in parallel branches. The transport delay introduced by this wideband filter is 6.5 sample intervals.

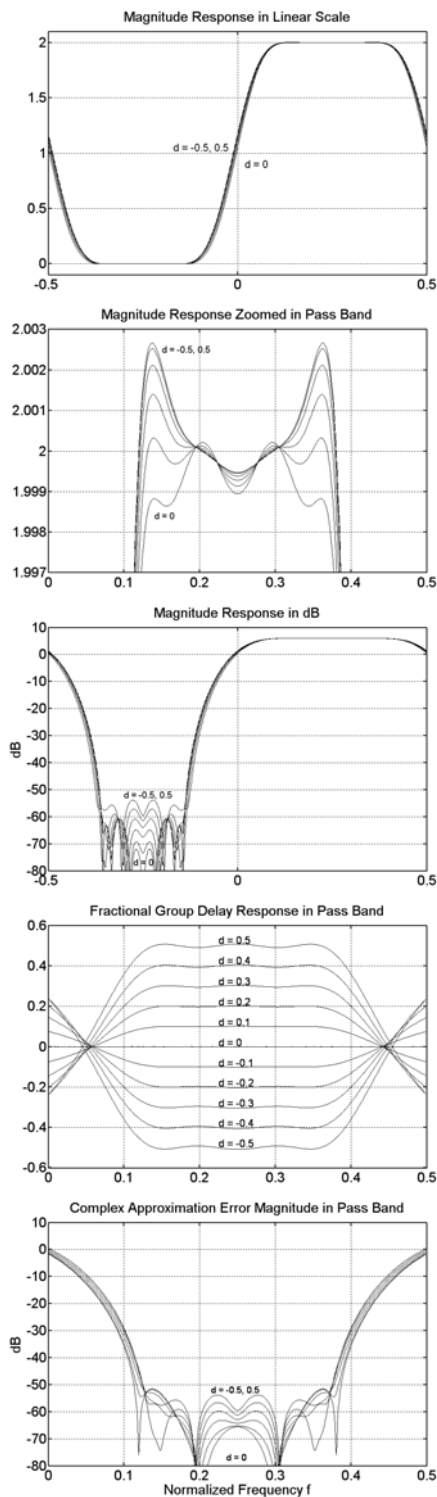


Fig. 5. The performance of an aggregated minimax HTF-VFD of length 14 with VFDs optimized in two-stage design.

The maximal value of the CAE magnitude of the resulting HTF-VFD (defined as the absolute value of the difference between the ideal frequency response (8) and its approximate) arranged as shown in Fig. 2 is approximately  $-51.5$  dB in the bandwidth of 0.225 located symmetrically

around the normalized frequency 0.25. Note that in case of  $d=0$ , the target HTF-VFD filter performance can be assessed in a way shown in [9], [10] or [11] on the basis of the performance of the VFD used in Fig. 2 as the VFDR.

## V. CONCLUSIONS

A novel efficient design of a digital filter for quadrature sub-sample delay estimation was proposed and described in this paper. The coefficients of this filter are obtained from the coefficients of a pair of identical VFDs appropriately “rotated” and interlaced. Maximally flat (Lagrangian) [6] approximation as well as minimax approximation [7], [8] of VFDs optimized in two-stage design was employed to compute coefficients of the designed HTF-VFD filter. Using in this design the structure invented by Farrow [4] gives a possibility of changing the value of introduced delay in real-time, without redesigning the coefficients. The original Farrow structure [4] is composed of linear-phase subfilters and as such can be used straightforwardly without any need for modifications. (This seems unnoticed by many authors.) It was shown that the proposed aggregated HTF-VFD exhibits wider bandwidth and introduces smaller integer delay in comparison with a hitherto applied cascade design based on a series connection of two independent filters: the linear-phase HTF and the VFD.

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